Josephson effect in strongly disordered metallic wires

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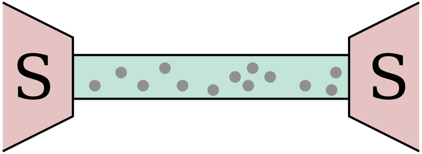
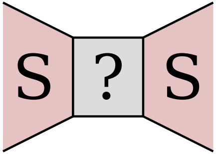


Les Houches, 9 June 2023

Outline

Beenakker formula

[Beenakker '91]



- Generic Josephson junction with an arbitrary scatterer
- Assumptions:
  - Superconducting leads with the order parameter Δ
  - Set of transmission eigenvalues T_i
  - Time-reversal symmetry in the normal state
  - "Short" junction: T_i energy independent (on the scale Δ)
- Current-phase relation (Beenakker formula)

$$I(\phi) = \frac{e\Delta}{2\hbar} \sum_i \frac{T_i \sin\phi}{\sqrt{1 - T_i \sin^2(\phi/2)}}$$

Ambegaokar-Baratoff relation

[Ambegaokar, Baratoff '63]

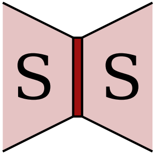
- Tunnel junction between superconductors
- All transmission probabilities small $T_i \ll 1$
- Normal conductance (Landauer formula):

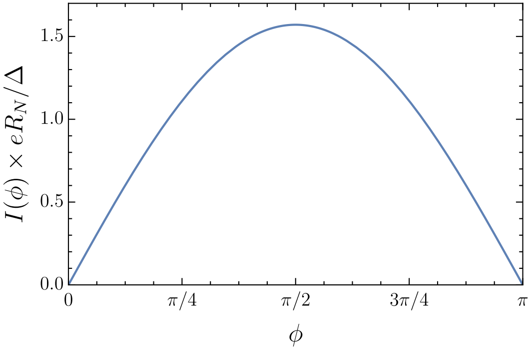
$$\frac{1}{R_N} = \frac{e^2}{\pi\hbar} \sum_i T_i$$

- Current-phase relation:

$$I(\phi) = \frac{e\Delta}{2\hbar} \sum_i T_i \sin\phi = \frac{\pi\Delta}{2eR_N} \sin\phi$$

- General AB relation: $I_c R_N \sim \Delta/e$

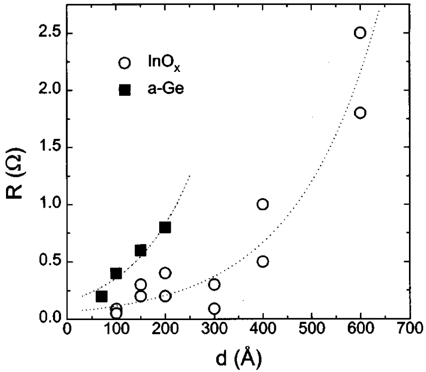


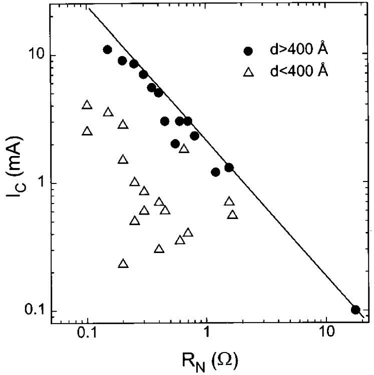


Experimental motivation

[Frydman, Ovadyahu '97]

Long junctions with exponentially suppressed conductance





Still obey Ambegaokar-Baratoff!

Short diffusive SNS

[Kulik, Omelyanchuk '75]

- Normal diffusive wire $l \ll L \ll \xi,\ E_{\mathrm{Th}} \gg \Delta$
- Classical distribution of T [Dorokhov '84]

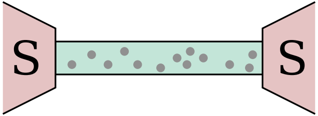
$$\rho(T) = \frac{\pi\hbar}{2e^2 R_N} \frac{1}{T\sqrt{1-T}}$$

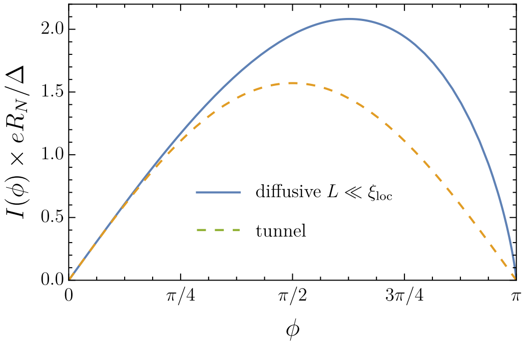
- Apply Beenakker formula

$$I(\phi) = \frac{\pi\Delta}{4eR_N} \int_0^1 \frac{dT\,\sin\phi}{\sqrt{1-T}\sqrt{1-T\sin^2(\phi/2)}}$$

$$= \frac{\pi\Delta}{eR_N} \operatorname{arcsinh}\left(\tan\frac{\phi}{2}\right)\cos\frac{\phi}{2}$$

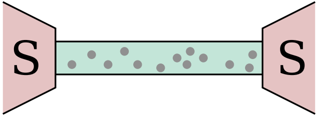
Ambegaokar-Baratoff relation holds





Usadel equation

[Usadel '70]



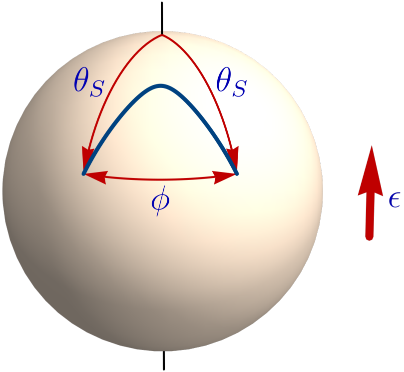
- Quasiclassical equation for hybrid structures
- Classical motion on a sphere in a uniform force field

$$S = \frac{\pi\nu}{2}\int dx \left[D\left(\frac{\partial\theta}{\partial x}\right)^2 + D\sin^2\theta\left(\frac{\partial\phi}{\partial x}\right)^2 - 4\epsilon\cos\theta \right]$$

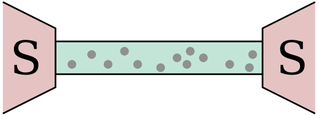
- Coordinate x along the wire plays the role of time
- Boundary conditions with superconductor

$$\theta = \theta_S = \arctan\frac{\Delta}{\epsilon}, \quad \phi = \phi_S$$

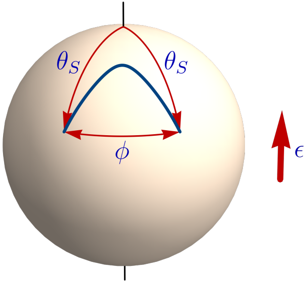
- Current $$I(\phi) = -\frac{2e}{\pi\hbar}\int_0^\infty d\epsilon \frac{\partial S_{\mathrm{min}}}{\partial\phi}$$



Short vs. long junction

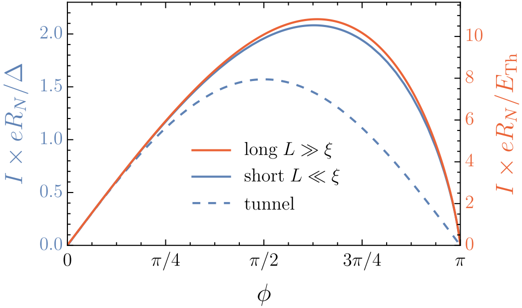


$$S = \frac{\pi\nu}{2}\int dx\left[D\left(\frac{\partial\theta}{\partial x}\right)^2 + D\sin^2\theta\left(\frac{\partial\phi}{\partial x}\right)^2 - 4\epsilon\cos\theta\right]$$



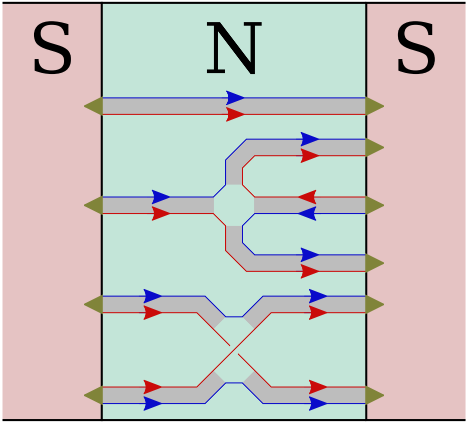
Two limiting cases:

- Short wire $L \ll \xi$, $\Delta \ll E_{\mathrm{Th}}$: Neglect ϵ in S reproduces [Kulik, Omelyanchuk '75]
  $$I(\phi) = \frac{\pi\Delta}{eR_N}\,\mathrm{arcsinh}\left(\tan\frac{\phi}{2}\right)\cos\frac{\phi}{2}$$
- Long wire $L \gg \xi$, $\Delta \gg E_{\mathrm{Th}}$: Set $\theta_S = \pi/2$ numerical solution [Dubos *et al* '01]
  $$I(\phi) = \frac{E_{\mathrm{Th}}}{eR_N}\,\mathrm{function}(\phi)$$



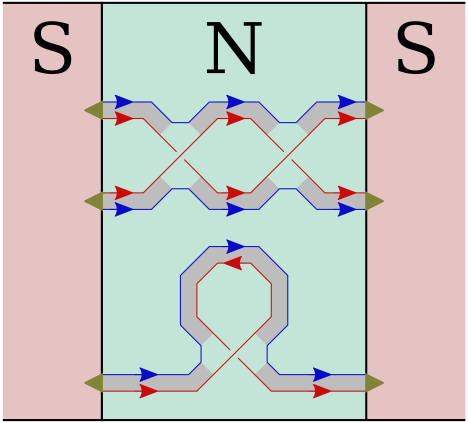
What about localization?

Quasiclassics (Usadel)



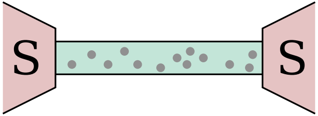
- Tree-like diagrams
- Only Cooperons

Localization



- Loop diagrams
- Cooperons and diffusons

Parameters of the problem

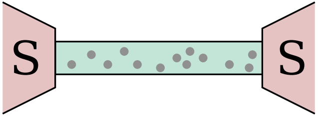


- Localization effects become important at long distances
- Typical scales:

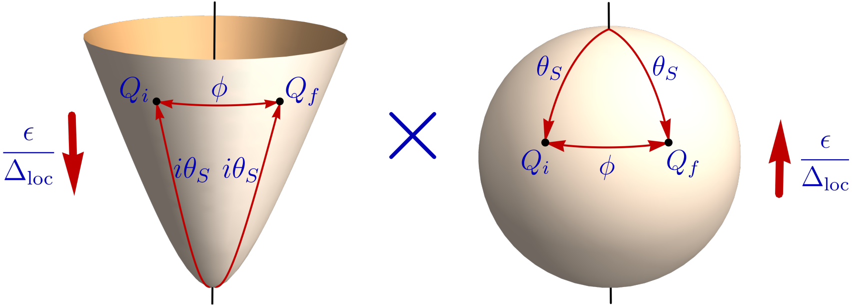
	Junction	Superconductivity	Localization
Length:	L	$\xi = \sqrt{D/\Delta}$	$\xi_{\text{loc}} = \pi\nu D = \sqrt{D/\Delta_{\text{loc}}}$
Energy:	$E_{\text{Th}} = D/L^2$	$\Delta = D/\xi^2$	$\Delta_{\text{loc}} = D/\xi_{\text{loc}}^2$

- Limiting cases
  - Classical limit (weak localization) $L \ll \xi_{\text{loc}}$. Solved using Usadel equation
    - $L \ll \xi$, $\Delta \ll E_{\text{Th}}$ classical short [Kulik, Omelyanchuk '75]
    - $L \gg \xi$, $\Delta \gg E_{\text{Th}}$ classical long [Dubos *et al* '01]
  - Quantum limit (strong localization) $L \gg \xi_{\text{loc}}$. To be solved using sigma model

Nonlinear sigma model

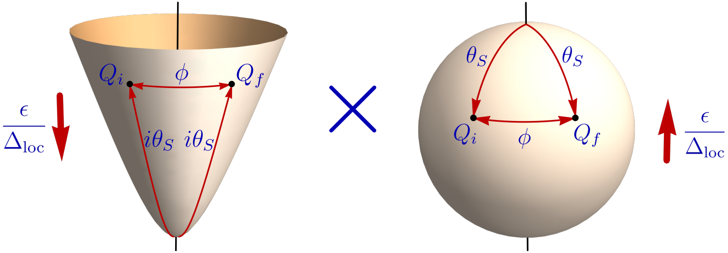


- Sigma model action $S = -\frac{\pi\nu}{8}\int_0^L dx\,\mathrm{str}\left[D(\nabla Q)^2 - 4\epsilon\Lambda Q\right]$
- Quantum dynamics on a curved superspace in fictitious time x
- Hamiltonian: $H = -\nabla_Q^2 + \frac{\epsilon}{\Delta_{\text{loc}}}\,\mathrm{str}(\Lambda Q)$
- Boundary conditions $Q_{i,f} \Leftrightarrow \{\theta_S, \phi\}$, $\tan\theta_S = \Delta/\epsilon$
- Structure of the manifold $\approx$ product of hyperboloid (bosons) and sphere (fermions):



How to solve?

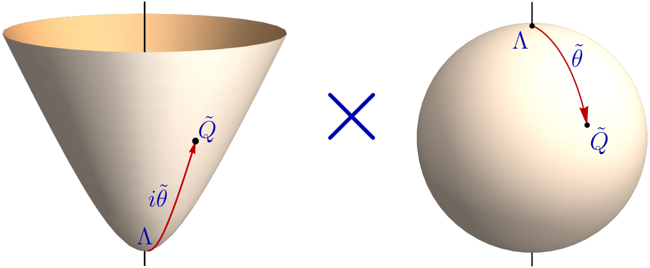
1. Find eigenstates of the Hamiltonian: $\left[-\nabla_Q^2 + \frac{\epsilon}{\Delta_{\text{loc}}} \operatorname{str}(\Lambda Q)\right] \psi_\alpha(Q) = E_\alpha \psi_\alpha(Q)$
2. Compute evolution operator $W = \sum_\alpha \psi_\alpha(Q_i)\psi_\alpha(Q_f) \exp[-E_\alpha L/\xi_{\text{loc}}]$
3. Take derivative in phase $I(\phi) = -\frac{2e}{\pi\hbar} \int_0^\infty d\epsilon \left. \frac{\partial W}{\partial \phi_F} \right|_{\phi_{B,F}=\phi}$



- $L \gg \xi_{\text{loc}}$: consider only lowest eigenstates
- $\Delta \ll \Delta_{\text{loc}}$: treat potential perturbatively

Quantum wire $L \gg \xi_{\text{loc}}$

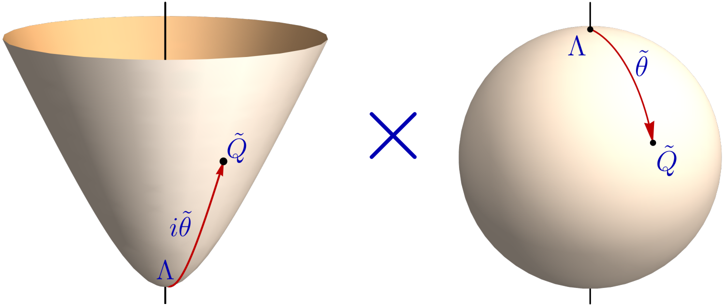
- Disregard potential term completely $\Longrightarrow$ full rotational symmetry
- Rotate $Q = T_S^{-1}\tilde{Q}T_S$ to bring $Q_i \mapsto \Lambda$



- Boundary condition: $\cos\{\tilde{\theta}_F, i\tilde{\theta}_B\} = \cos^2\theta_S + \sin^2\theta_S \cos\phi_{B,F} = \dfrac{\epsilon^2 + \Delta^2\cos\phi_{B,F}}{\epsilon^2 + \Delta^2}$

Only eigenstates invariant in $\tilde{\phi}$ are involved!

Evolution operator



- $\tilde{\phi}$-invariant eigenstates ($l = 0, 1, 2, 3, \ldots,\ 0 < q < +\infty$):

$$\psi_{l,q} = (\cosh\tilde{\theta}_B - \cos\tilde{\theta}_F) P_l(\cos\tilde{\theta}_F) P_{-1/2+iq}(\cosh\tilde{\theta}_B), \qquad E_{l,q} = (l + 1/2)^2 + q^2$$

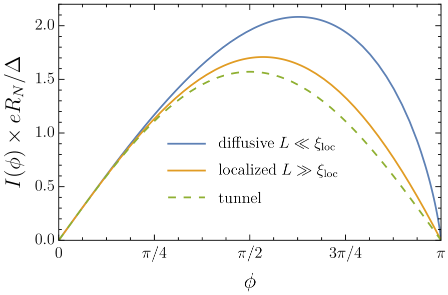
- Evolution operator at $L \gg \xi_{\mathrm{loc}}$ (take $l = 0$ and $q \ll 1$):

$$W(\tilde{\theta}_B, \tilde{\theta}_F) \approx 1 - \pi^2 (\cosh\tilde{\theta}_B - \cos\tilde{\theta}_F) K\left(\cosh\frac{\tilde{\theta}_B}{2}\right) \int_0^\infty dq q^2 e^{-(q^2+1/4)L/\xi_{\mathrm{loc}}}$$

$$= 1 - \frac{\pi^{5/2}\xi_{\mathrm{loc}}^{3/2}}{4L^{3/2}} e^{-L/4\xi_{\mathrm{loc}}} (\cosh\tilde{\theta}_B - \cos\tilde{\theta}_F) K\left(\cosh\frac{\tilde{\theta}_B}{2}\right)$$

Current-phase relation

- Normal conductance: $\frac{1}{R_N} = -\frac{2e^2}{\pi\hbar} \frac{\partial^2 W}{\partial \tilde{\theta}_F^2}\Big|_{\tilde{\theta}_{B,F}=0} = \frac{\pi e^2}{4\hbar}\left(\frac{\pi\xi_{\mathsf{loc}}}{L}\right)^{3/2} e^{-L/4\xi_{\mathsf{loc}}}$
- Energy enters only via boundary conditions: $\cos\{\tilde{\theta}_F, i\tilde{\theta}_B\} = \frac{\epsilon^2 + \Delta^2\cos\phi_{B,F}}{\epsilon^2+\Delta^2}$
- Current-phase relation: $I(\phi) = -\frac{2e}{\hbar}\int_0^\infty \frac{d\epsilon}{\pi}\frac{\partial W}{\partial \phi_F}\Big|_{\phi_{B,F}=\phi} = \frac{2\Delta}{\pi e R_N} K^2\left(\sin\frac{\phi}{4}\right)\sin\phi$



Ambegaokar-Baratoff relation holds!

Effect of potential term I

- Restore small potential term (relevant only on hyperboloid)

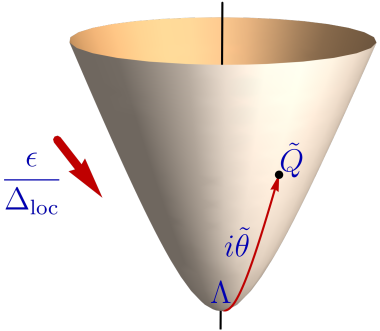
$$H = -\frac{\partial^2}{\partial\tilde{\theta}^2} - \coth\tilde{\theta}\frac{\partial}{\partial\tilde{\theta}} - \frac{1}{\sinh^2\tilde{\theta}}\frac{\partial^2}{\partial\tilde{\phi}^2} + \frac{\epsilon}{\Delta_{\mathrm{loc}}}\cosh\theta$$

- Potential effectively introduces a wall at $\theta_{\mathrm{max}} \sim \ln\frac{\Delta_{\mathrm{loc}}}{\epsilon} \gg 1$
- Rotational symmetry is broken but...

⟨⟨⟨a lengthy calculation skipped⟩⟩⟩

...the value of q is quantized $q_n = \frac{\pi n}{\ln(\Delta_{\mathrm{loc}}/\epsilon)}$

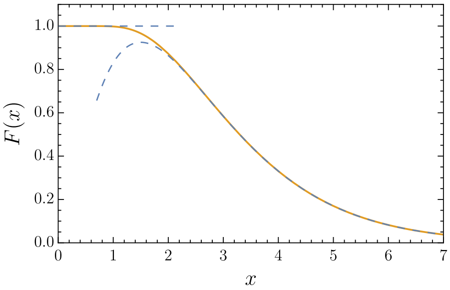
- At $L \sim \xi_{\mathrm{loc}}\ln^2(\Delta_{\mathrm{loc}}/\epsilon)$, discreteness becomes important

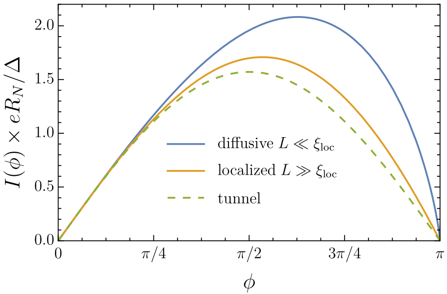


Effect of potential term II

- Quantization $q_n = \dfrac{\pi n}{\ln(\Delta_{\text{loc}}/\epsilon)} \quad \Longrightarrow \quad$ replace q-integral by n-sum
- Extra factor in the current $I(\phi) = \dfrac{2\Delta}{\pi e R_N} F\left(\dfrac{\pi^2 L}{\xi_{\text{loc}} \ln^2(\Delta_{\text{loc}}/\Delta)}\right) K^2\left(\sin\dfrac{\phi}{4}\right) \sin\phi$
- Function $F(x)$ quantifies suppression of the Ambegaokar-Baratoff relation

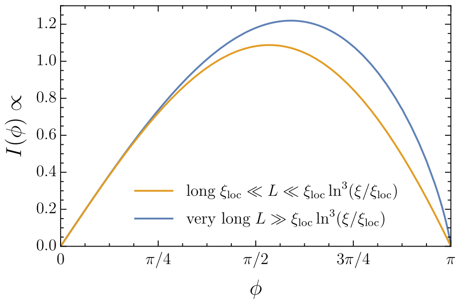
$$F(x) = \frac{4}{\sqrt{\pi}} x^{3/2} \sum_{n=1}^{\infty} n^2 e^{-xn^2} \approx \begin{cases} 1, & x \ll 1 \\ \dfrac{4}{\sqrt{\pi}} x^{3/2} e^{-x}, & x \gg 1 \end{cases}$$





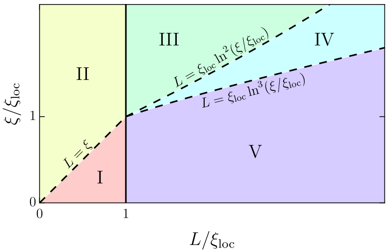
Very long junction $L \gg \xi_{\text{loc}} \ln^3(\Delta_{\text{loc}}/\Delta)$

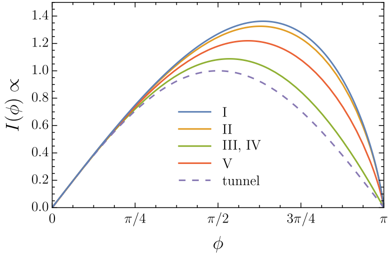
- At a length $L \gg \xi_{\text{loc}} \ln^2(\Delta_{\text{loc}}/\Delta)$ only the lowest eigenstate with $q = \dfrac{\pi}{\ln(\Delta_{\text{loc}}/\epsilon)}$ contributes
- Spectral integral $I(\phi) = -\dfrac{2e}{\pi\hbar}\displaystyle\int_0^\infty d\epsilon \left.\dfrac{\partial W}{\partial \phi_F}\right|_{\phi_{B,F}=\phi}$ is dominated by the region $\epsilon \lesssim \Delta$
- At yet longer distances $L \gg \xi_{\text{loc}} \ln^3(\Delta_{\text{loc}}/\Delta)$ spectral integral comes from $\epsilon \ll \Delta$
- Supercurrent $I(\phi) = \dfrac{\pi^3 E_{\text{Th}}}{\sqrt{3}\, e R_N}\left(\dfrac{2L}{\pi\xi_{\text{loc}}}\right)^{8/3} \exp\left[-\dfrac{3\pi}{2}\left(\dfrac{2L}{\pi\xi_{\text{loc}}}\right)^{1/3}\right] K\left(\sin\dfrac{\phi}{2}\right)\sin\phi$



Summary of results

	Critical current	Current-phase relation
I	$\dfrac{E_{\mathsf{Th}}}{eR_N}$	$\text{function}(\phi)$ [Usadel]
II	$\dfrac{\Delta}{eR_N}$ [Ambegaokar-Baratoff]	$\operatorname{arcsinh}\left(\tan\dfrac{\phi}{2}\right)\cos\dfrac{\phi}{2}$
III	$\dfrac{\Delta}{eR_N}$ [Ambegaokar-Baratoff]	$K^2\left(\sin\dfrac{\phi}{4}\right)\sin\phi$
IV	$\dfrac{\Delta}{eR_N}\left(\dfrac{L}{\xi_{\mathsf{loc}}\ln^2(\xi/\xi_{\mathsf{loc}})}\right)^{3/2}\exp\left[-\dfrac{2.5L}{\xi_{\mathsf{loc}}\ln^2(\xi/\xi_{\mathsf{loc}})}\right]$	$K^2\left(\sin\dfrac{\phi}{4}\right)\sin\phi$
V	$\dfrac{E_{\mathsf{Th}}}{eR_N}\left(\dfrac{L}{\xi_{\mathsf{loc}}}\right)^{8/3}\exp\left[-4.1\left(\dfrac{L}{\xi_{\mathsf{loc}}}\right)^{1/3}\right]$	$K\left(\sin\dfrac{\phi}{2}\right)\sin\phi$





Summary

1. Complete theory of Josephson effect in 1D SNS junctions with full account of localization
2. Current-phase relation in five different asymptotic regimes
3. Ambegaokar-Baratoff relation holds far beyond localization length (regions II and III)