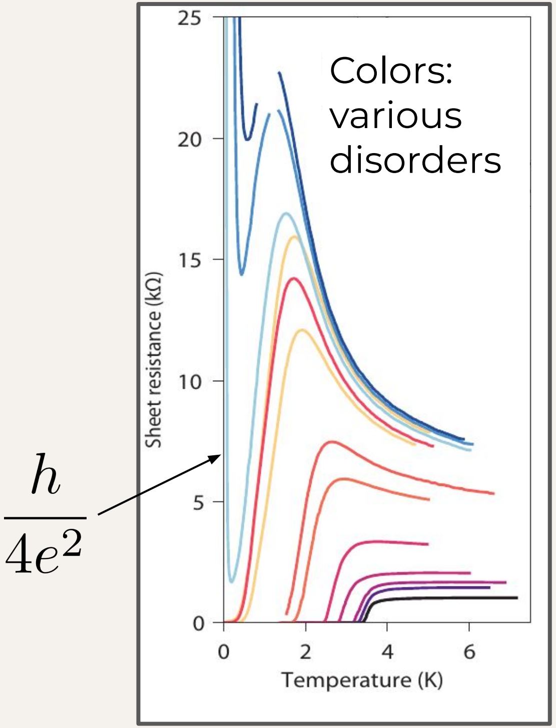
Temperature dependence of the Superfluid stiffness in strongly disordered superconductors

Anton Khvalyuk and Mikhail Feigelman

Contents of this talk

01
Phenomenology and experimental motivation

Insulating trend above T_c



Experimental facts:

1. Activation law in normal state

$$R_\square(T) = R_0 \exp\{-T_1/T\}, \quad T > T_c$$

2. No semiclassical limit

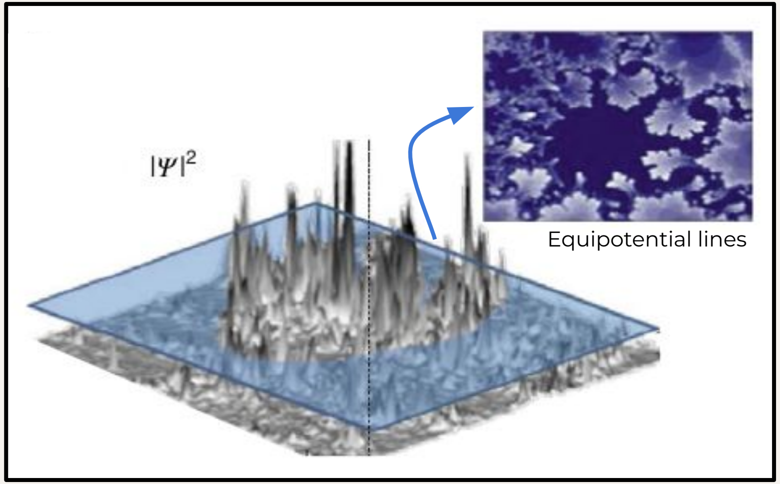
$$\frac{4e^2}{h} R_\square \propto \frac{1}{k_F l} \gtrsim 1$$

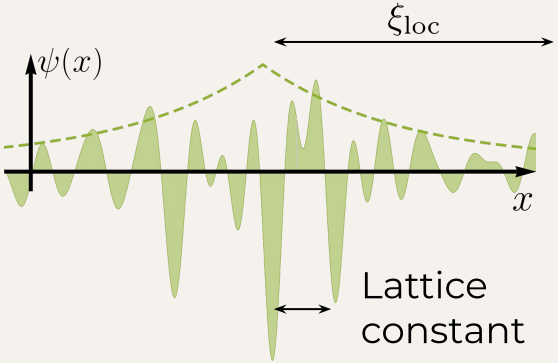
3. T_c strongly depends on disorder =>
   no Anderson theorem (in agreement with 2)

[0] T. Charpentier et al, to be published.

Anderson insulator 101

Numerics: fractal wave function close to delocalization [1]





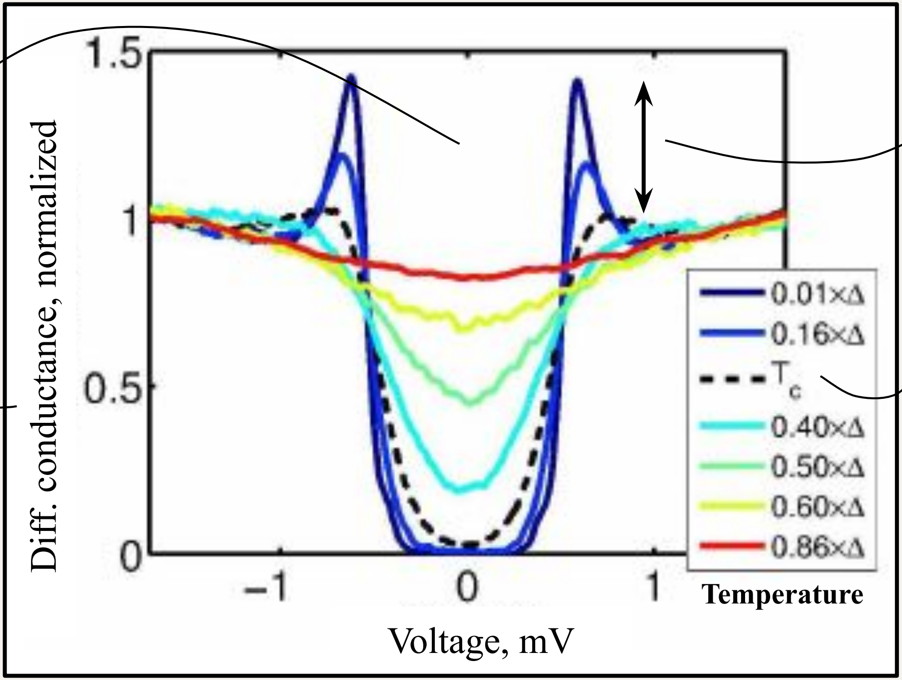
- **Localized** single-particle states
- **No diffusion**, insulating behavior
- **Fractal** wave functions

[1] Courtesy of Vladimir Kravtsov; see also B.Sacépé, Nat. Phys. 2020

No quasiparticle excitations

Hard gap
even above T_c,
Not order parameter

∝ single-particle DoS

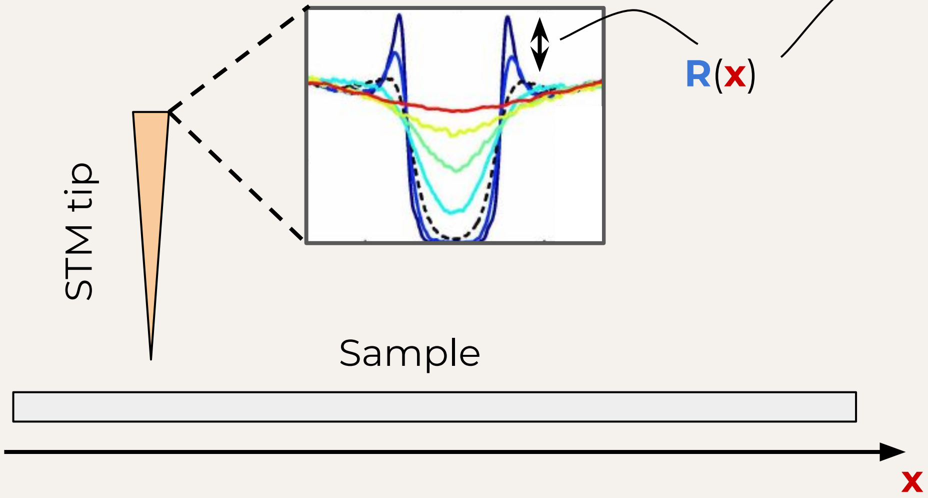


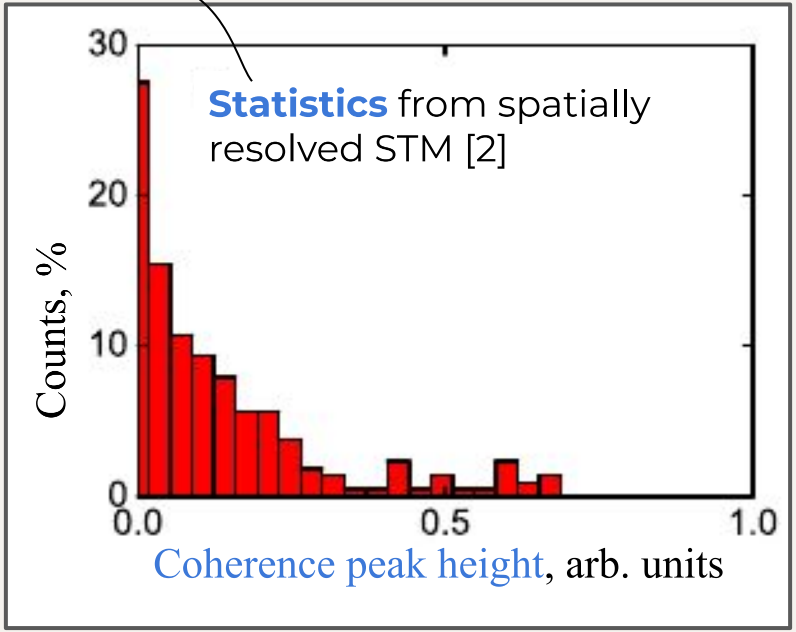
Coherence peak height is a proxy to local “order parameter”

Determined from resistance

[2] B.Sacépé et al, Nature Physics 2011

Inhomogeneous superconducting state





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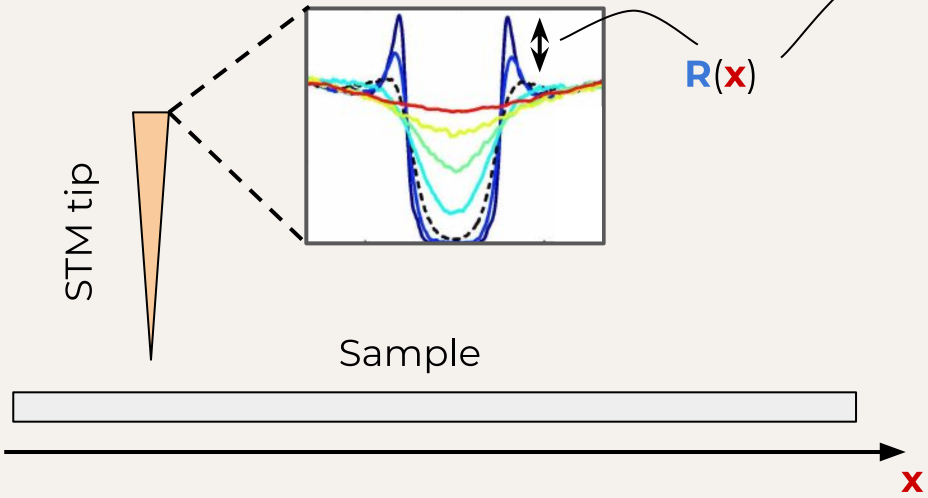
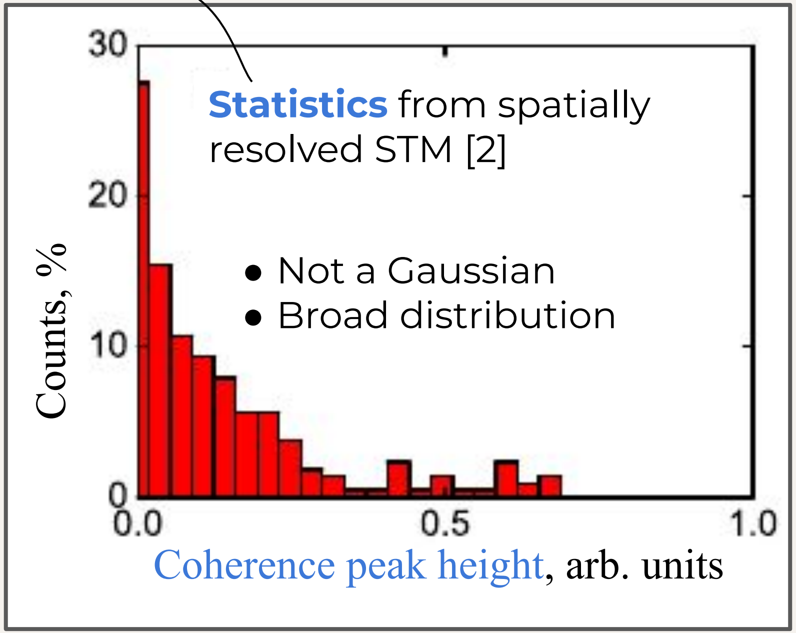


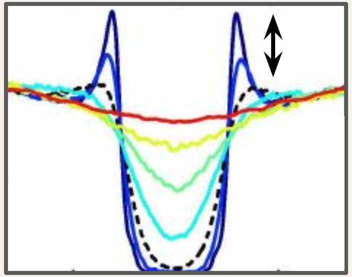


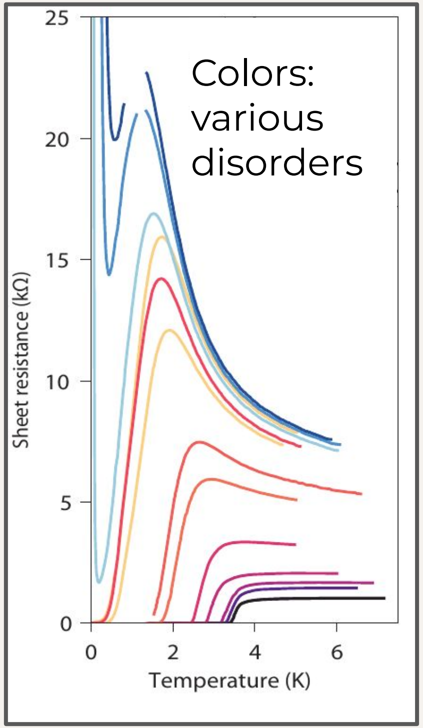
[2] B.Sacépé et al, Nature Physics 2011

Theoretical explanation of experiments [3]

T < T_c: Superconductor

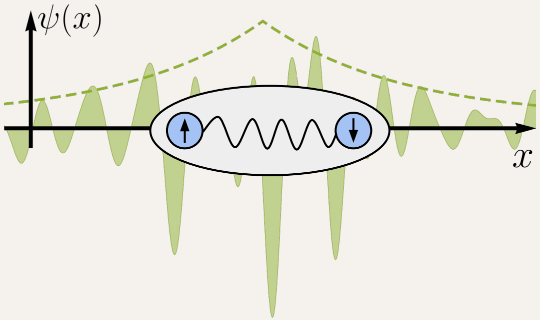
- Cooper pairs delocalize due to attraction
- Global phase coherence
- Quasiparticles are still gapped





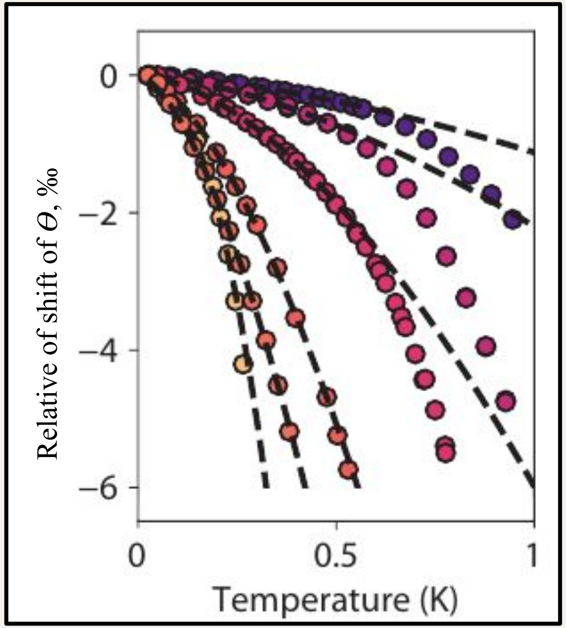
T > T_c: not a metal

- Preformed cooper pairs
- Localized in single-particles state
- Large binding energy, quasiparticles are gapped
- No phase coherence



[3] M.V. Feigel'man et al, Annals of Phys. 2010

Temperature dependence of the superfluid stiffness [0]



Experimental facts:

1. Dependence is not exponential. $\cancel{\Theta(T) - \Theta(0) \propto \exp\left\{-\frac{\Delta}{T}\right\}}$

2. Power law (dashed lines) describes the low-T data better. $\frac{\Theta(T) - \Theta(0)}{\Theta(0)} = -\left(T/T_0\right)^b$

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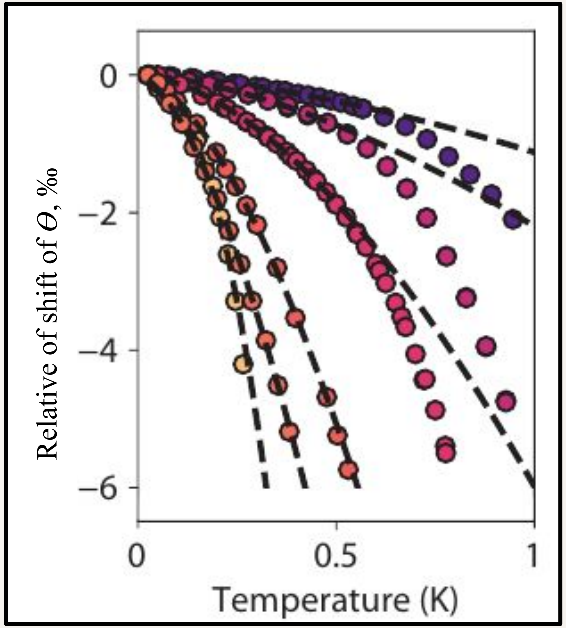

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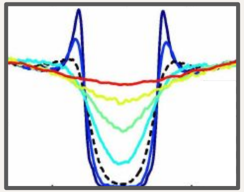
Implications:

- Excitation DoS is nonzero,
- but those are not quasiparticles.

$$\frac{\delta\Theta}{\Theta} \sim -\int_0^\infty d\epsilon\, \nu(\epsilon)\, e^{-\epsilon/T}$$



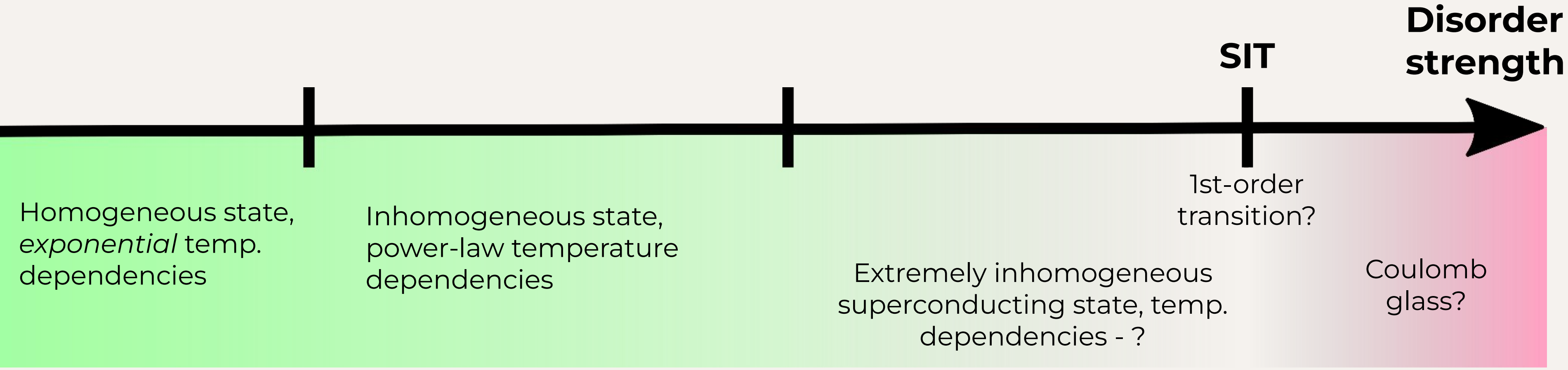
What excitations suppress θ?

[0] T. Charpentier et al, to be published.

03

Model Hamiltonian

Relevant range of parameters [5]



We are here

... and nor here ...

... neither here.

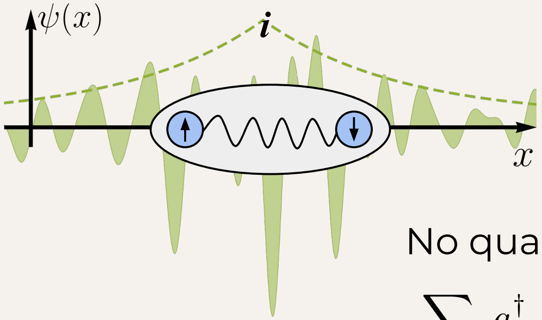
[5] A.V. Khvalyuk and M.V. Feigel'man, PRB 2021

Cooper pairs in localized single-particle states [3,4]

$$H = -\sum_i \sum_{\sigma=\uparrow,\downarrow} \xi_i a^\dagger_{i\sigma} a_{i\sigma}$$

Index of localized single-particle state

Random energies, DoS $\nu(\xi)$, scale E_F



No quasiparticles:

$$\sum_{\sigma=\uparrow,\downarrow} a^\dagger_{i\sigma} a_{i\sigma}\ |\text{state}\rangle = \{0 \text{ or } 2\}\ |\text{state}\rangle$$

We throw away part of the Hilbert space with single occupancy numbers

[3] M.V. Feigel'man et al, Annals of Phys. 2010; [4] M.V. Feigel'man and L.B. Ioffe, PRL 2018

Cooper pairs in localized single-particle states [3,4]

$$H = -\sum_i 2\xi_i S_i^z$$

$a^\dagger_{i\sigma} a_{i\sigma}$

Index of localized single-particle state

Random energies, DoS $\nu(\xi)$, scale E_F

Pseudospin operators

$$S_i^z = \frac{1}{2}\left(\sum_{\sigma=\uparrow,\downarrow} a^\dagger_{i\sigma} a_{i\sigma} - 1\right)$$

$$S_i^+ = a^\dagger_{i\downarrow} a^\dagger_{i\uparrow},\ S_i^- = a_{i\uparrow} a_{i\downarrow}$$

[3] M.V. Feigel'man et al, Annals of Phys. 2010; [4] M.V. Feigel'man and L.B. Ioffe, PRL 2018

Hopping of Cooper pairs in localized single-particle states [3,4]

$$H = -\sum_i 2\xi_i S_i^z - \sum_{\langle ij\rangle} 4J_{ij}\left(S_i^x S_j^x + S_i^y S_j^y\right)$$

$$-a_{i\uparrow}^\dagger a_{i\downarrow}^\dagger J_{ij} a_{j\downarrow} a_{j\uparrow}$$

Energy dependence at the scale of ω_D

Wave function overlap

$$J_{ij} = D\left(|\xi_i - \xi_j|\right) \int d^3\boldsymbol{r}\, |\psi_i(\boldsymbol{r})|^2 \, |\psi_j(\boldsymbol{r})|^2$$

Phonon-induced short-range interaction

[3] M.V. Feigel'man et al, Annals of Phys. 2010; [4] M.V. Feigel'man and L.B. Ioffe, PRL 2018

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$$-a_{i\uparrow}^{\dagger} a_{i\downarrow}^{\dagger} J_{ij} a_{j\downarrow} a_{j\uparrow}$$

Superconductivity:

$$\Delta_i \propto \langle a_{\downarrow i} a_{\uparrow i}\rangle = \left\langle S_i^- \right\rangle \neq 0$$

Position-dependent!

Energy dependence at the scale of ω_D

Wave function overlap

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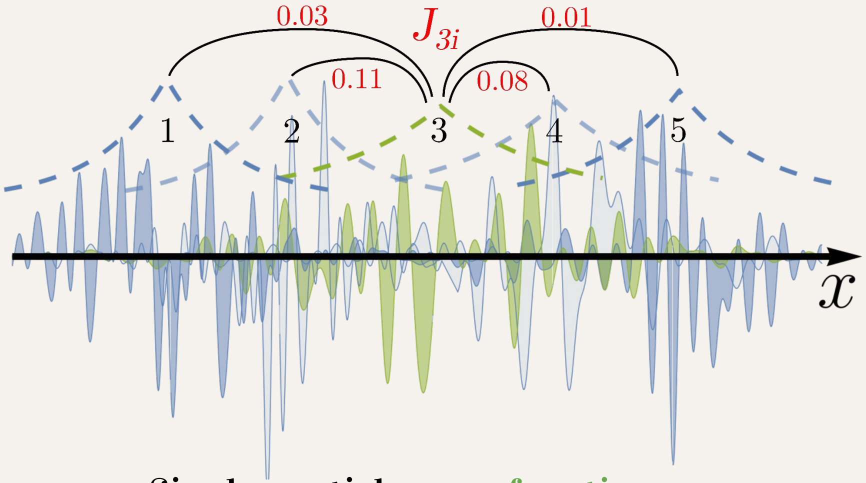
Approximation for the matrix elements [3,4]

Cooper attraction matrix elements:



Single-particle wave functions

$$J_{ij} = D\left(|\xi_i - \xi_j|\right) \int d^3\boldsymbol{r}\; |\psi_i(\boldsymbol{r})|^2\, |\psi_j(\boldsymbol{r})|^2$$

- Decay with distance r_{ij}
- Fluctuate strongly

[3] M.V. Feigel'man et al, Annals of Phys. 2010; [4] M.V. Feigel’man and L.B. Ioffe, PRL 2018

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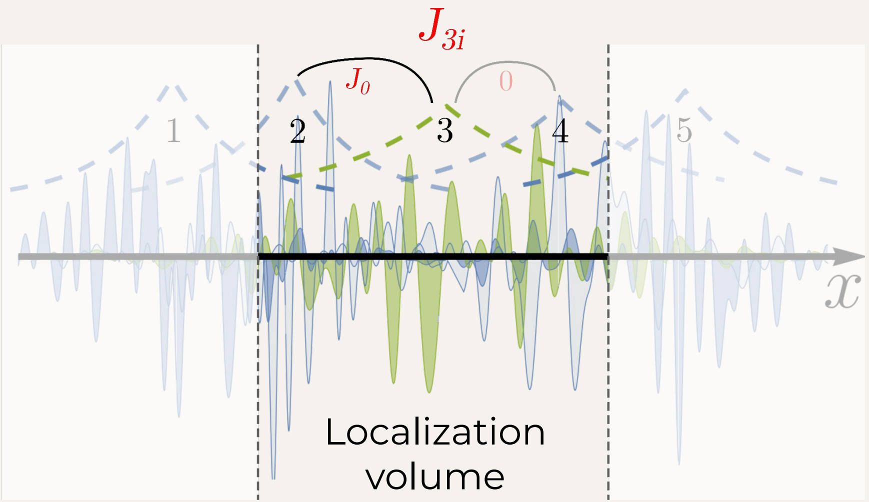

Dimensionless Cooper coupling constant

$$J_{ij} = \begin{cases} J_0 = \frac{\lambda n}{2\nu_0 K}, & \text{"strong"} \\ 0 & \text{"weak"} \end{cases}$$

Simplest model:

- Neighbors chosen randomly in the localization volume
- *Constant* number of neighbors *K + 1*

[3] M.V. Feigel'man et al, Annals of Phys. 2010; [4] M.V. Feigel'man and L.B. Ioffe, PRL 2018

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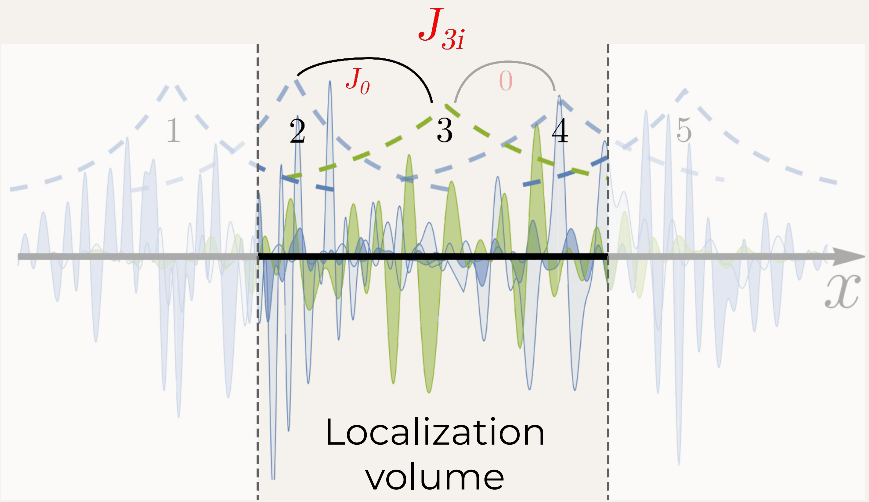


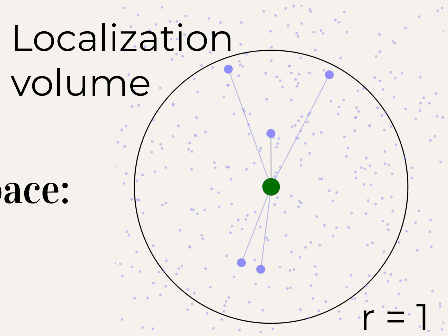
$$J_{ij} = \begin{cases} J_0 = \frac{\lambda n}{2\nu_0 K}, & \text{"strong"} \\ 0 & \text{"weak"} \end{cases}$$

Main feature: locally tree-like structure of J_{ij}
notion of a **graph** embedded in real space

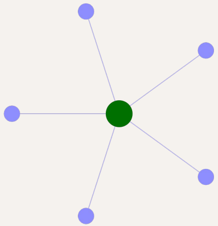
[3] M.V. Feigel'man et al, Annals of Phys. 2010; [4] M.V. Feigel'man and L.B. Ioffe, PRL 2018

Embedding of a locally tree-like structure of the matrix elements

Real space:



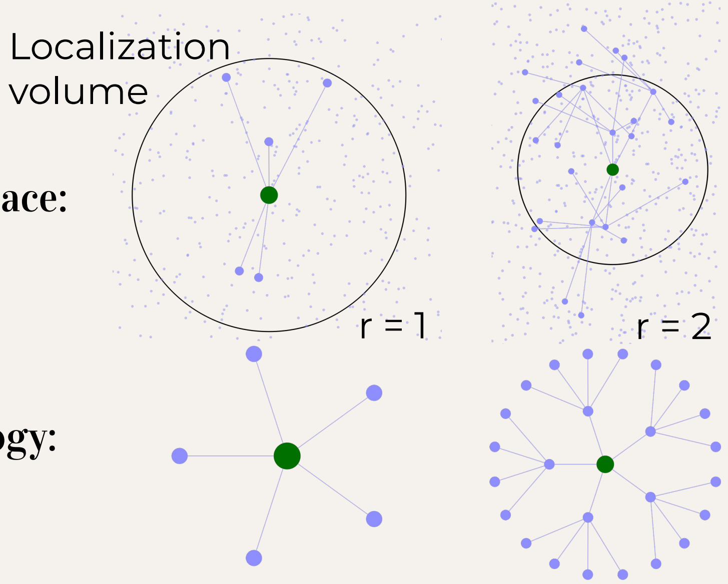
Topology:



Embedding of a locally tree-like structure of the matrix elements

Real space:

Topology:

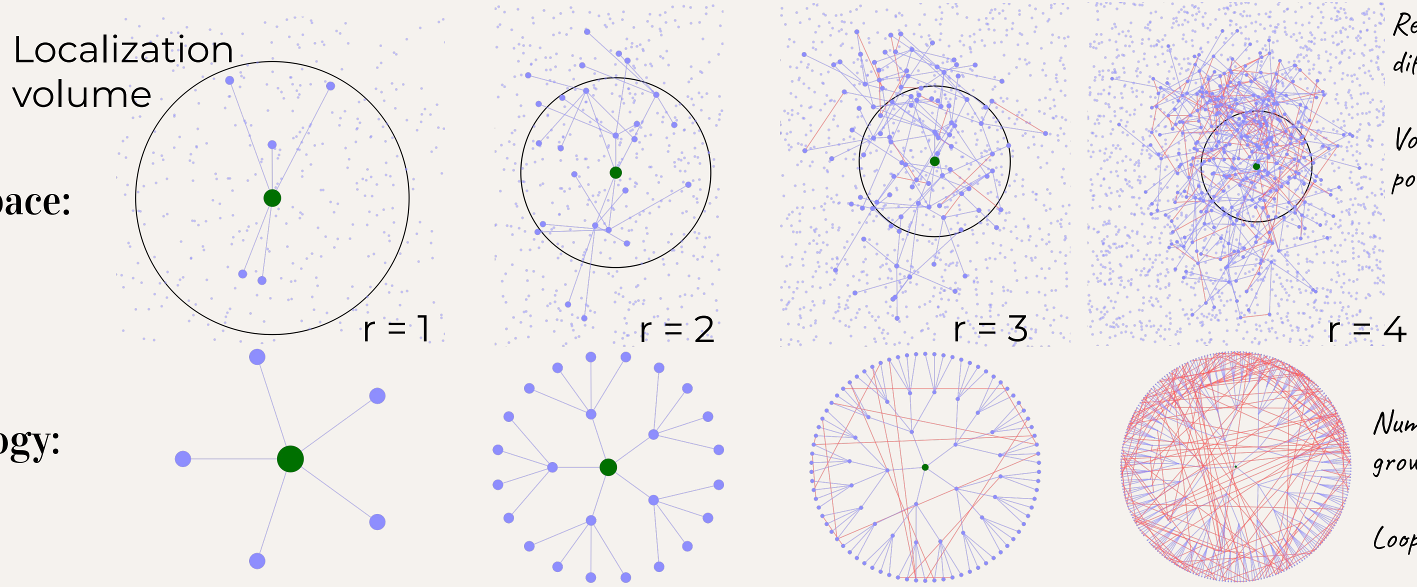


This we call the (portion of) interaction graph

Embedding of a locally tree-like structure of the matrix elements

Real space:

Topology:



Real distance grows diffusively with r

Volume grows as some power of r

Number of neighbors grows exponentially with r

Loops condense

Choice of units: $n = 1, \quad 2\nu_0 = 1$
Distance Energy

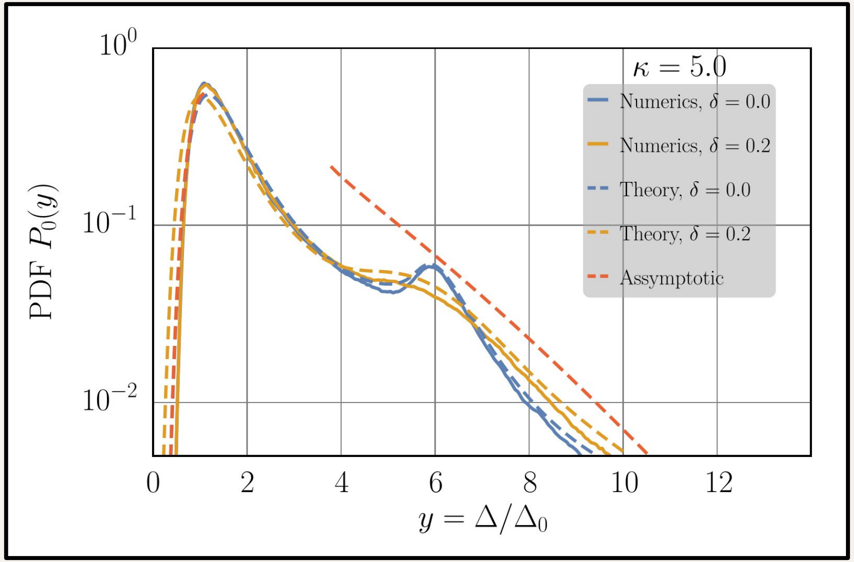
A bit of theory: distribution of pairing amplitude [5]

Superconducting Scale of energies: $\Delta_0 = 2e^{-1/\lambda}$

Dimensionless disorder strength $\kappa = \lambda / K\Delta_0$

Universal* and nontrivial distrib. of the pairing ampl. →

- **Broad** distribution
- Result of **nonlinearity** of equations
- Most features model-independent



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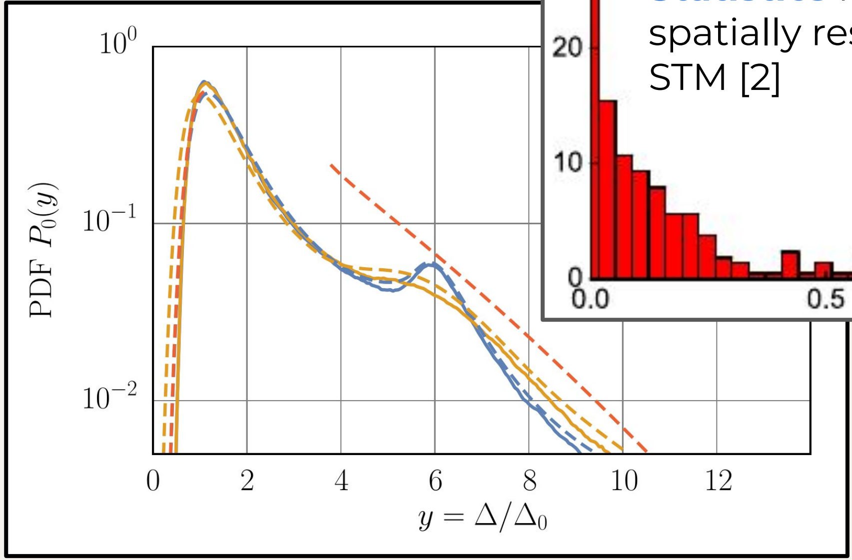

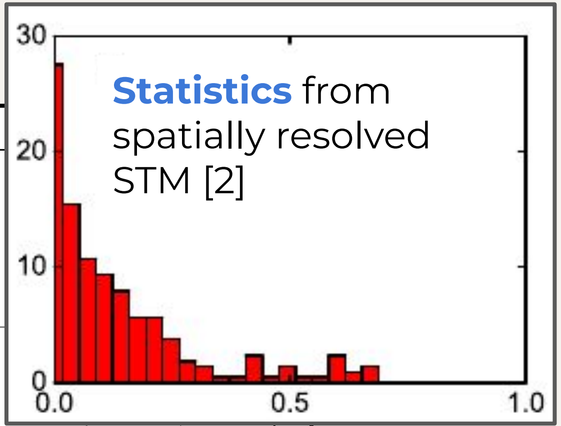

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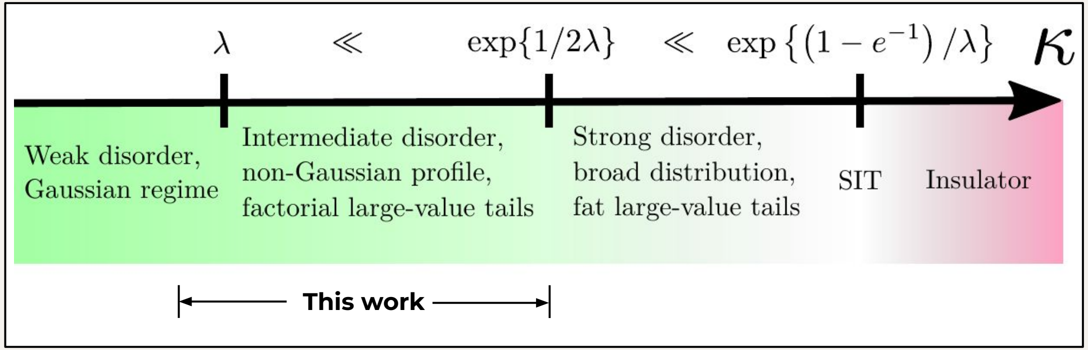

[5] A.V. Khvalyuk and M.V. Feigel'man, PRB 2021;

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Dimensionless disorder strength $\kappa = \lambda / K\Delta_0$

Universal* and nontrivial distrib. of the pairing ampl.

No direct connection between Δ and physical observables

$$\Theta d = \cancel{\frac{1}{(2e)^2}\frac{\pi\Delta}{R_\square}\tanh\frac{\Delta}{2T}}$$

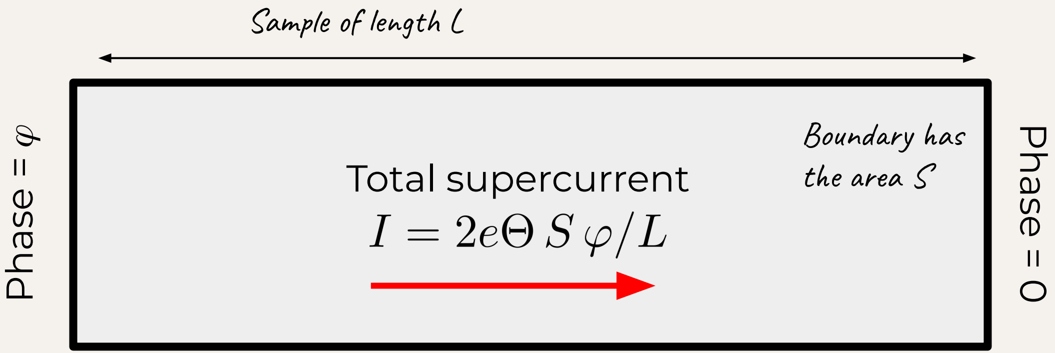
What Δ one should even use?
What is normal conductivity for an insulator?

03

Superfluid response of a disordered superconductor

Macroscopic superfluid stiffness

$$\boldsymbol{j} = 2e\Theta \left(\boldsymbol{\nabla}\phi - \frac{2e}{c}\boldsymbol{A} \right)$$



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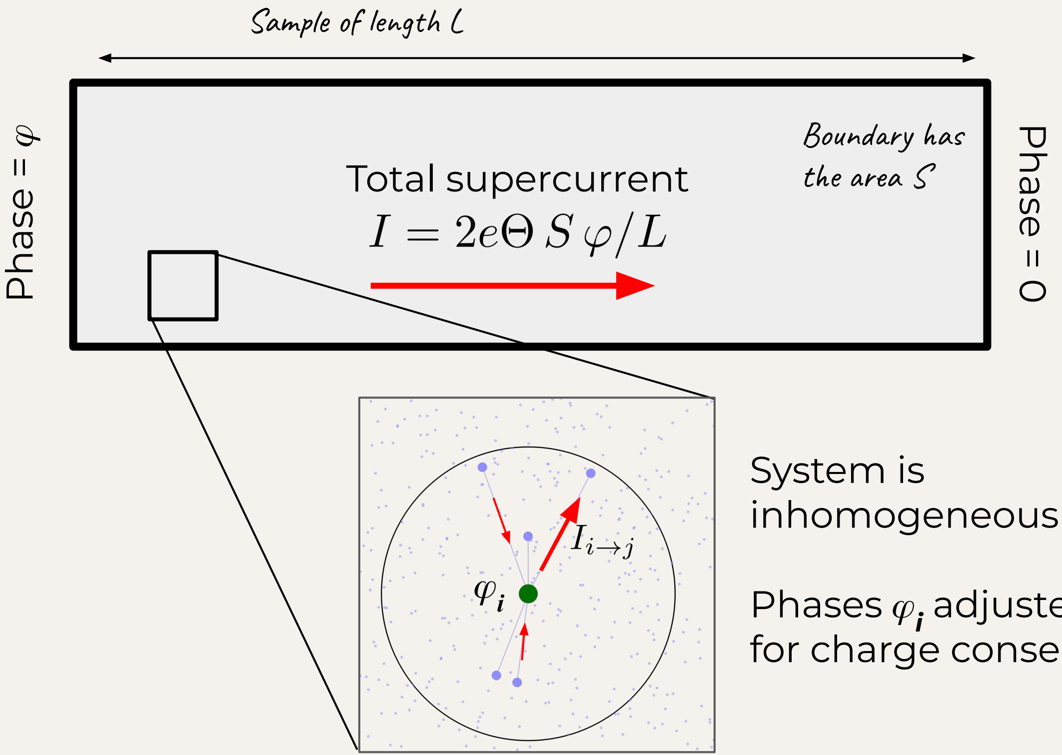

System is inhomogeneous

Phases φ_i adjusted for charge conserv.

Macroscopic superfluid stiffness

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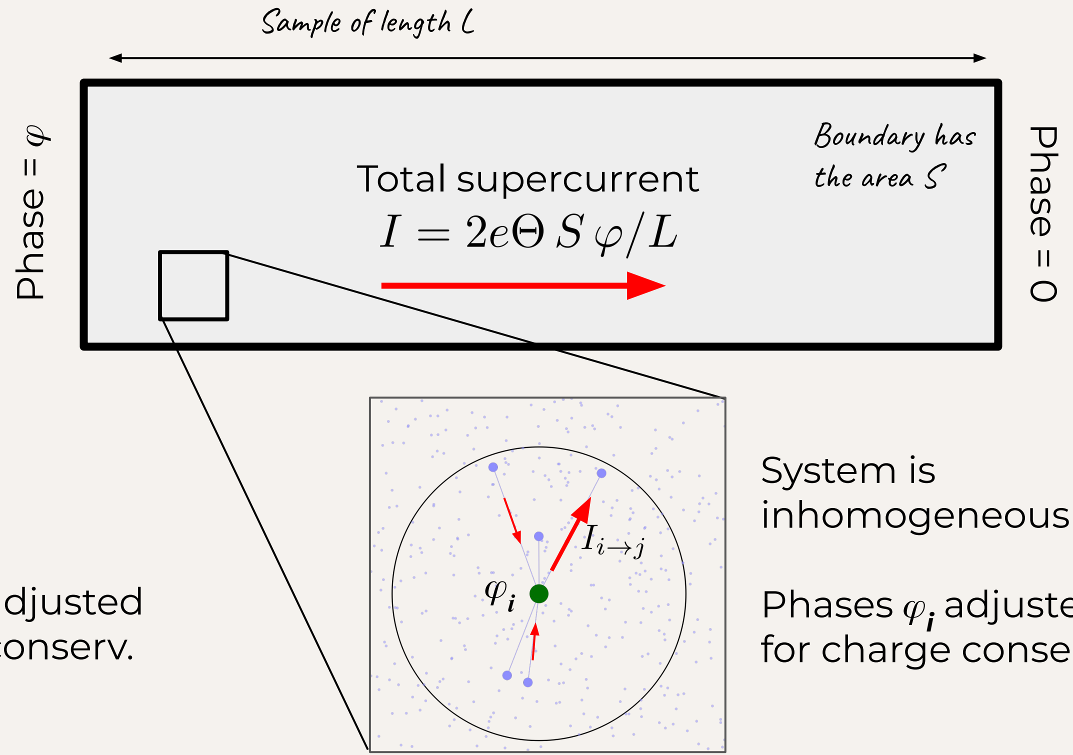

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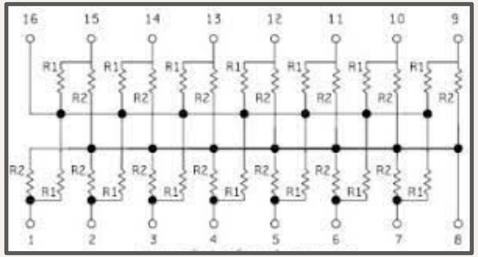
Phases φ_i adjusted for charge conserv.

Formal analogy: network of resistors:



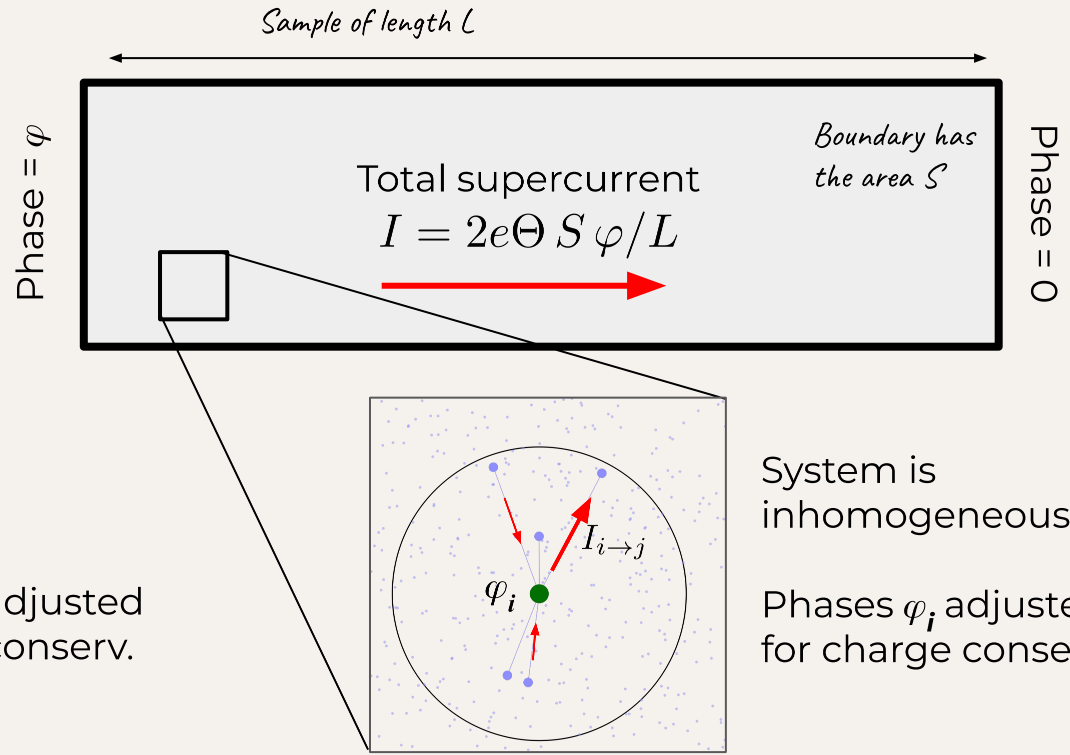
Potentials adjusted for charge conserv.

Macroscopic superfluid stiffness

Kirchoff's law

$$\begin{cases} \sum_{j\in\partial i} I_{i\to j} = 0 \\ I_{i\to j} = \frac{1}{2e} Q_{ij} \cdot (\varphi_j - \varphi_i) \\ \sum_{\text{boundary}} I_{i\to j} = 2e\Theta\, S \frac{\varphi_{\text{right}} - \varphi_{\text{left}}}{L} \end{cases}$$

Discrete local Ohm's law



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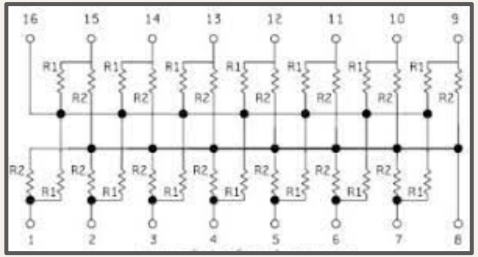

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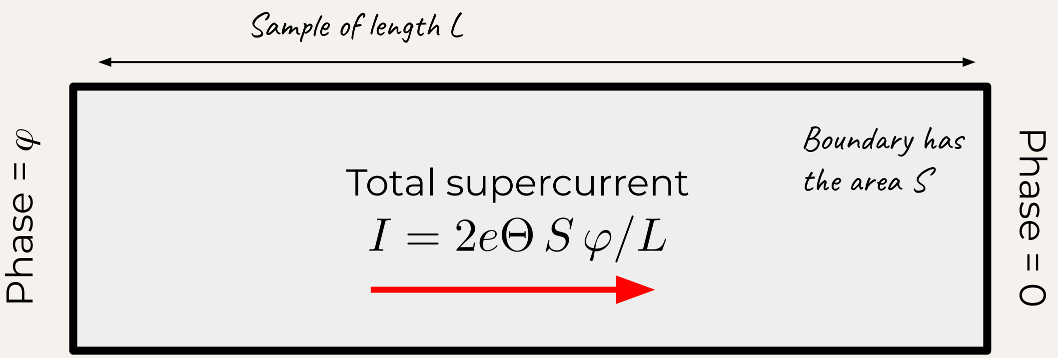
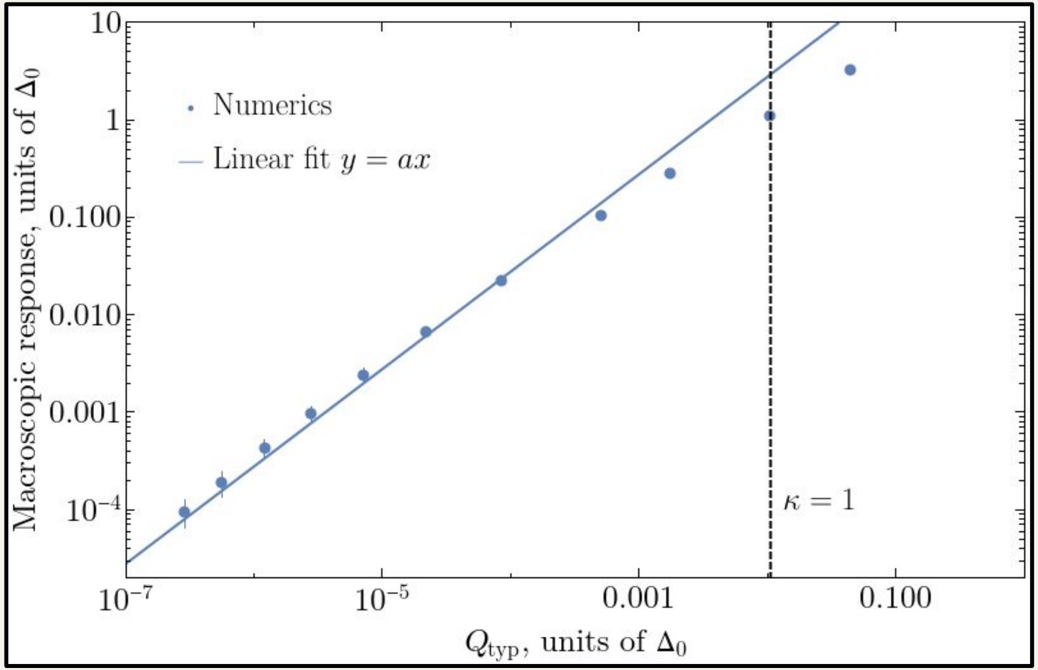
Note: just an illustration so far,
derivation: extra slides.



Q_{ij} are random (determined by microscopics),
the "topology" of the network is random,
the task is still nontrivial!

“Solving” the problem on the graph



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Numerics:

T-indep.

$$\ln \Theta\,(T) \approx A + B\, \ln Q_{\text{typ}}\,(T)$$

$$\ln Q_{\text{typ}}\,(T) \stackrel{\text{def}}{=} \overline{\ln Q_{ij}}$$

Sampled from microscopic theory

Kind of anticipated from [7]
(2D disordered conductor)

[7] A.M. Dykhne, JETP 1970

“Solving” the problem on the graph

Status: approximate law,
verified numerically

Q: how about 2D\3D?
A: did not test, but would not expect qualitative difference.

Q: can we derive the law for the graph?
A: there're ideas, it's a WIP.

Wanted: the statistics of Q_{ij}
from the Hamiltonian

Numerics:

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Disorder
average

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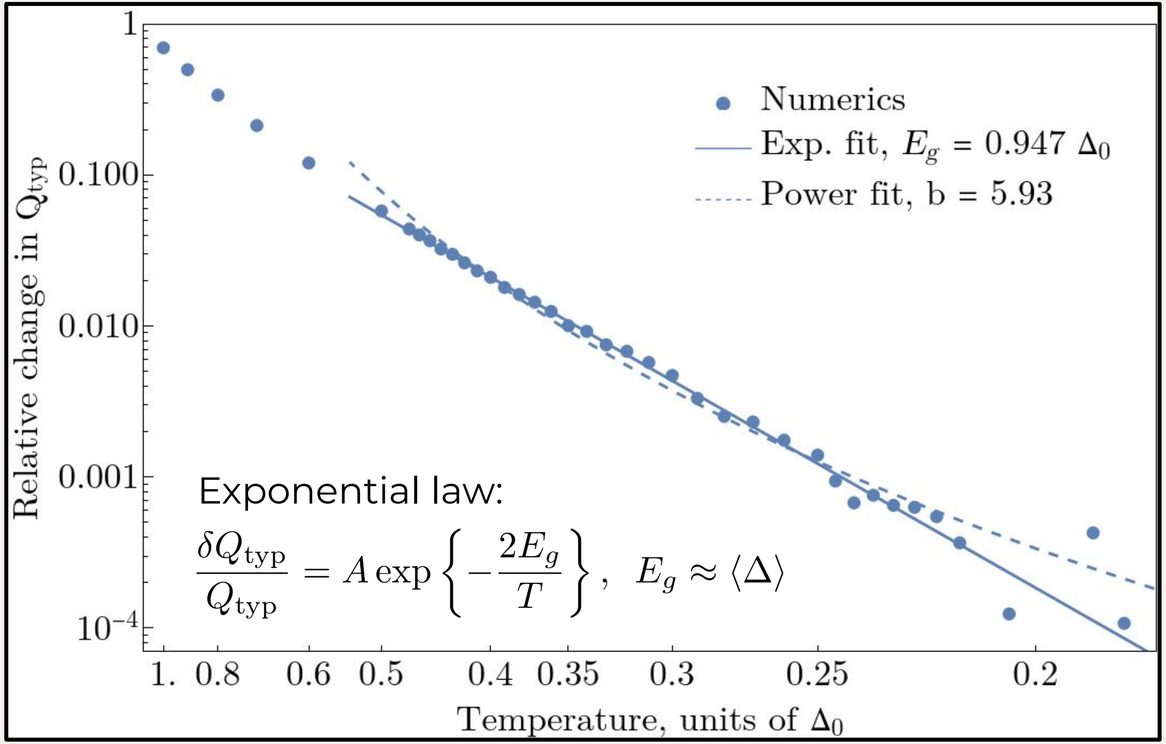
Disorder
average

Belief propagation on a tree, statistics of the
order parameter, analytical solution … **skipping**

04

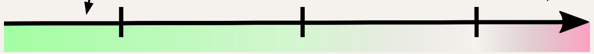
Results: theoretical low-T behavior of the superfluid stiffness

BCS-like behavior for weak disorder: $\kappa = 0.25$



Dimensionless disorder strength

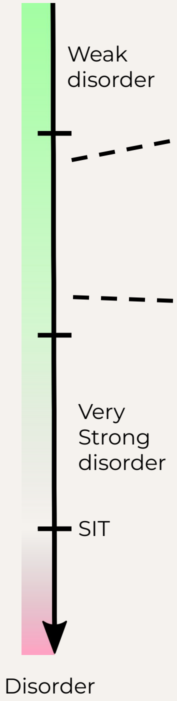
$\kappa = \lambda / K \Delta_0$

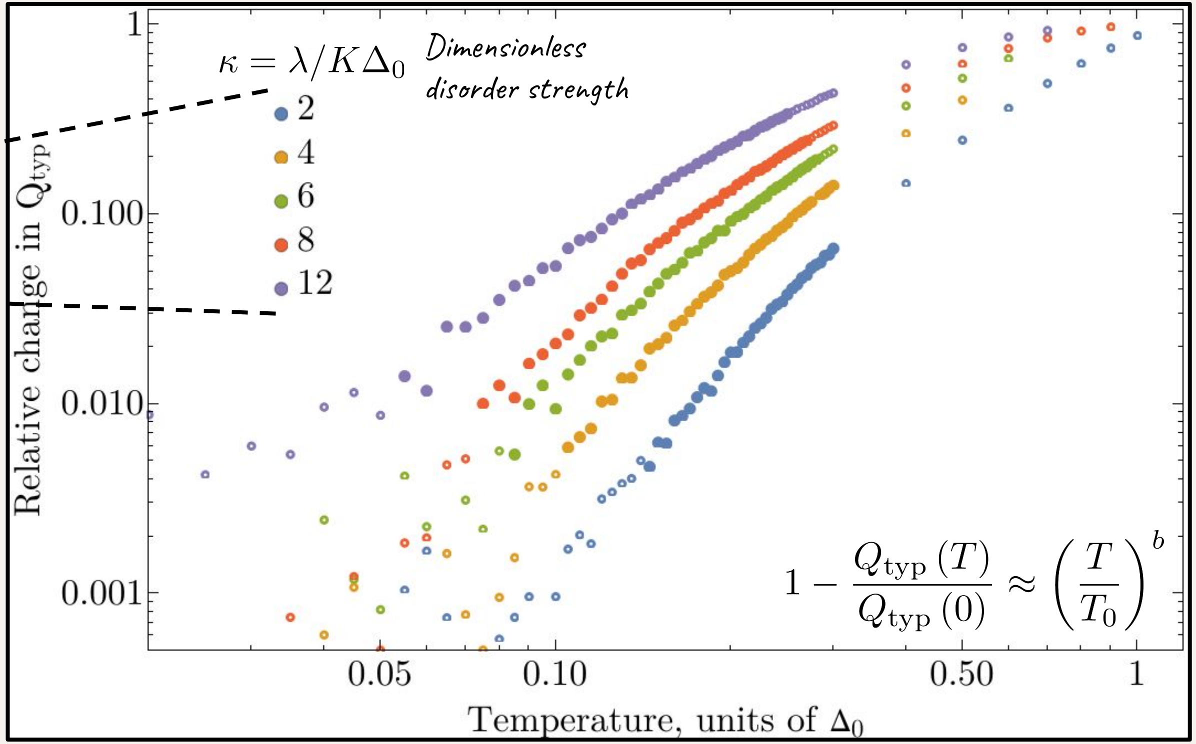


$$\ln \Theta(T) \approx A + B \ln Q_{\rm typ}(T)$$

$$\ln Q_{\rm typ}(T) = \overline{\ln Q_{ij}}$$

Results for strong disorder

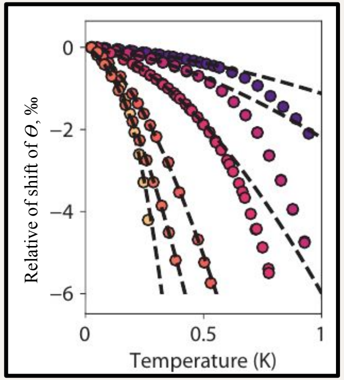




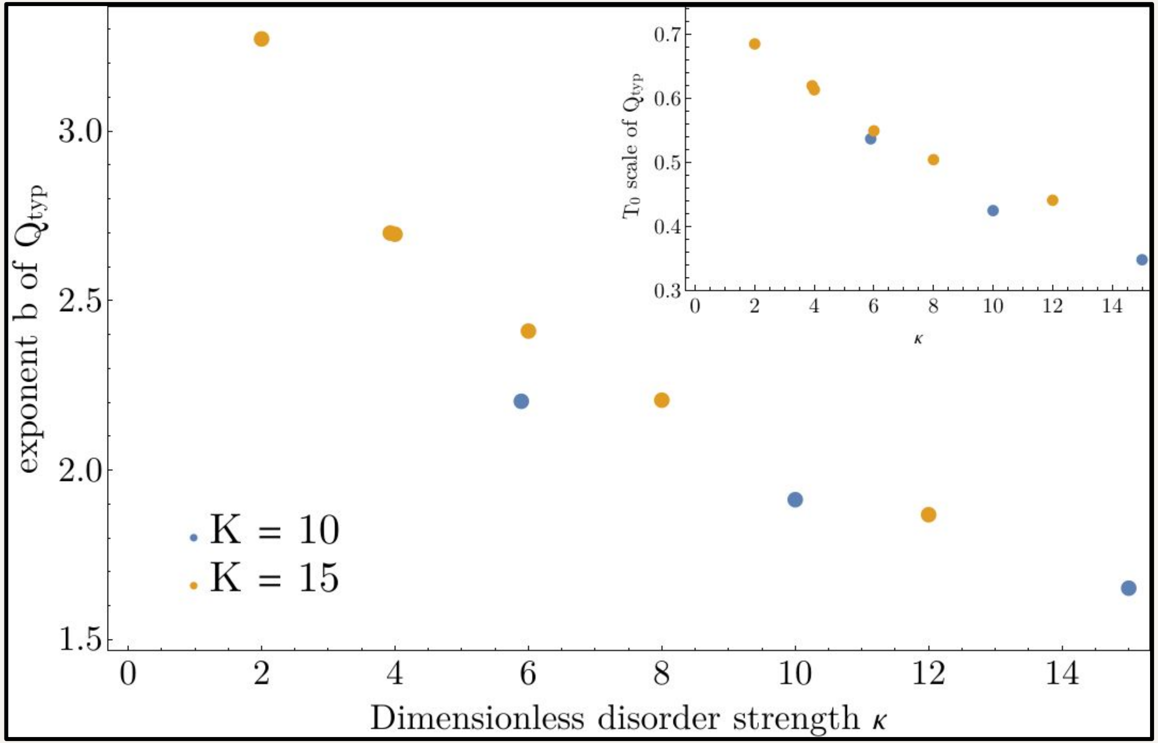
$$\ln \Theta(T) \approx A + B \ln Q_{\text{typ}}(T)$$

$$\ln Q_{\text{typ}}(T) = \overline{\ln Q_{ij}}$$

Observation:
Looks like power law, similar to experiment



Results for strong disorder

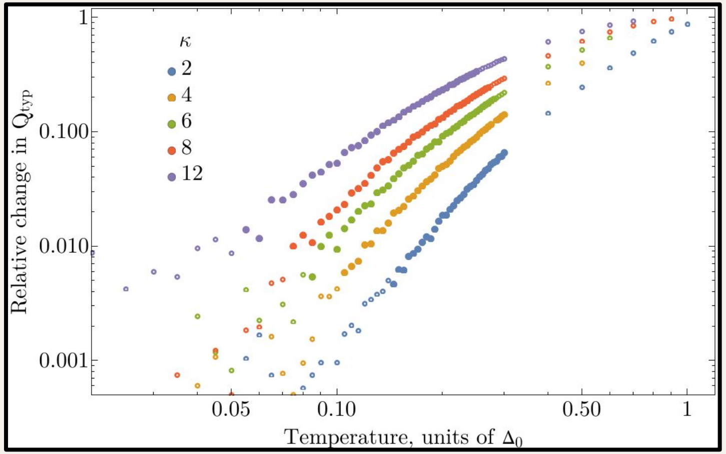


$$\ln \Theta(T) \approx A + B \ln Q_{\text{typ}}(T)$$

$$\ln Q_{\text{typ}}(T) = \overline{\ln Q_{ij}}$$

$$1 - \frac{Q_{\text{typ}}(T)}{Q_{\text{typ}}(0)} \approx \left(\frac{T}{T_0}\right)^b$$

One more peculiar thing:



For all highlighted points:

$$\frac{\delta\langle\Delta\rangle/\langle\Delta\rangle}{\delta Q_{\text{typ}}/Q_{\text{typ}}} = 0.5 \pm 0.05$$

Maybe not all that surprising:

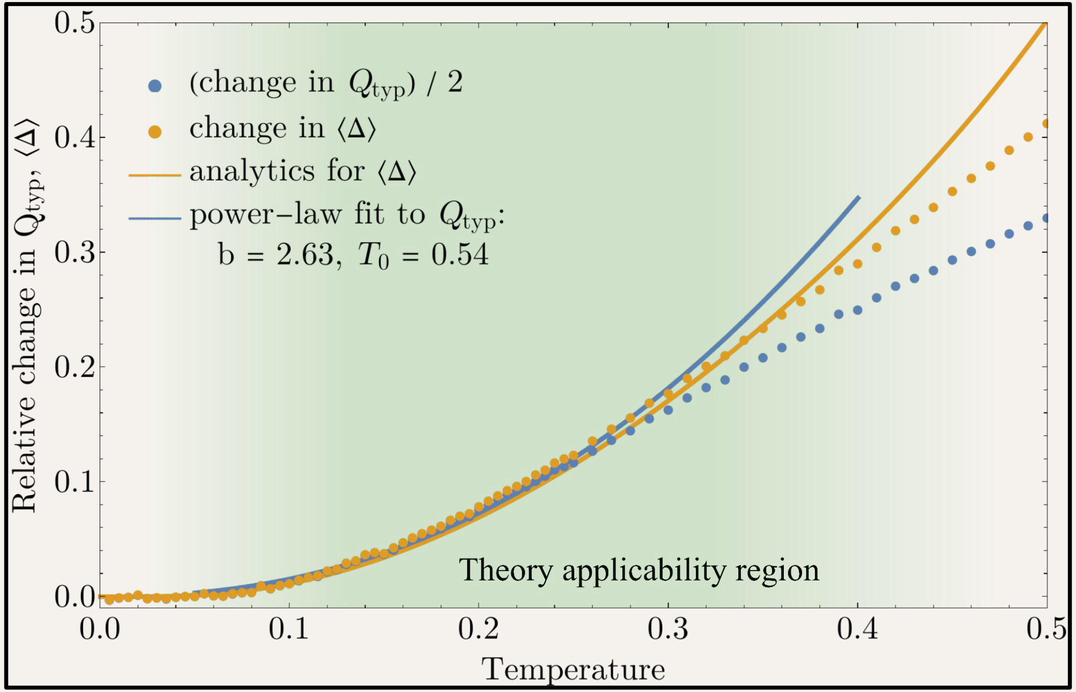
Ref. [8] $\Theta \propto \langle\Delta\rangle^2 \Rightarrow \frac{\delta\Theta}{\Theta} = 2\frac{\delta\langle\Delta\rangle}{\langle\Delta\rangle}$

Perturb. estim. $\frac{\delta Q_{\text{typ}}}{Q_{\text{typ}}} \approx \frac{\delta\Delta^2_{\text{typ}}}{\Delta^2_{\text{typ}}} \approx \frac{\delta\langle\Delta\rangle^2}{\langle\Delta\rangle^2} \approx 2\frac{\delta\langle\Delta\rangle}{\langle\Delta\rangle}$

We can estimate the change in Θ from that of Δ

[8] M.V. Feigel’man and L.B. Ioffe, PRB 2015

Analytical results for strong disorder



$$\ln\Theta(T) \approx A + B\ln Q_{\rm typ}(T)$$

$$\ln Q_{\rm typ}(T) = \overline{\ln Q_{ij}}$$

Roughly speaking,

From T=0

$$\delta\Theta \propto \delta\langle\Delta\rangle \propto \frac{\partial}{\partial\beta}\int_{2\beta e^{1-A/B}}^{1} dw\frac{\exp\left\{-A\frac{2\beta}{w} + B\frac{2\beta}{w}\ln\frac{2\beta}{w}\right\}}{\sqrt{1-w^2}}$$

Delivered by statistics of extremely low Δ

$K = 20,$
$\kappa = 10,$
$\lambda = 0.124$

[5] A.V. Khvalyuk and M.V. Feigel'man, PRB 2021

05
Conclusions

Recap:

Motivation: recent measurements

Model: tree-like pseudospin Hamiltonian

Microscopic maxwell equations

Ohm-like law, solution: Dykhne-like law (numerically)

Numerical answer of superfluid density *$\theta(T)$*

Numerical connection between θ and $\langle \Delta \rangle$

Analytical answer for θ(T)

Recap:

Motivation: recent measurements

Model: tree-like pseudospin Hamiltonian

Microscopic maxwell equations

Microscopic response identities

Ohm-like law, solution: Dykhne-like law (numerically)

Belief propagation, neglect quantum fluctuations

Numerical answer of superfluid density *$\Theta(T)$*

Numerical connection between Θ and $\langle \Delta \rangle$

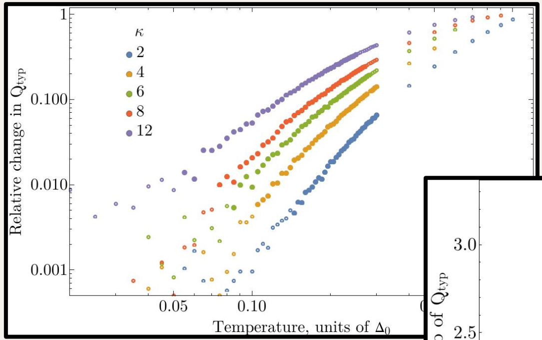
Analytical statistics of the order parameter Δ

Analytical answer for Θ(T)

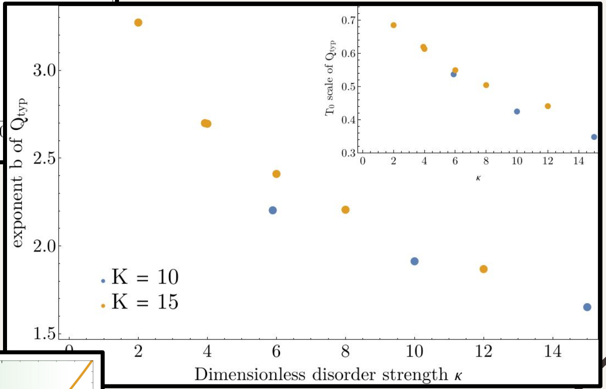
What are the excitations?

Our findings

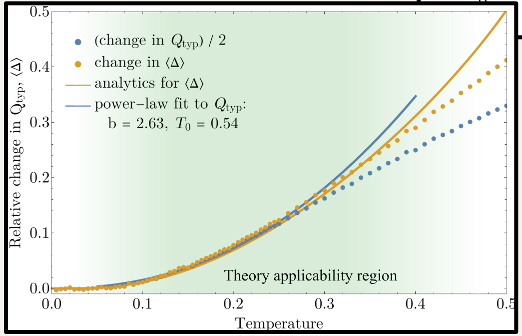
At moderately low temperatures,
The superfluid stiffness exhibits
power-law-like behavior



The character of the behavior
depends on disorder and
certain microscopics



The exact shape is dictated by the
extreme value statistics of the
order parameter



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[5] A.V. Khvalyuk and M.V. Feigel'man, PRB 2021

[6] M.V. Feigel’man et all, PRB 2010

[7] A.M. Dykhne, JETP 1970

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[9] J.S. Yedidia et al, Joint Conference on Artificial Intelligence, 2001

Thanks